



ANSWER BOOK

Arc length and surface area

Further Pure 2 · Year FM

7 questions · 32 marks

NAVY EDITION

SECTION A · Warm-up

- A1** $dy/dx = x^{1/2}$, so $1 + (dy/dx)^2 = 1 + x$ and the integrand is $\sqrt{1+x}$. Integrating [4]
gives $(2/3)(1+x)^{3/2}$, which runs from $2/3$ to $16/3$. The length is **14/3**, about 4.667. B1 for
 dy/dx , M1 for the arc length integrand, M1 for the integration, A1 for $14/3$.
- A2** $dy/dx = \sinh x$ and $1 + \sinh^2 x = \cosh^2 x$, so the integrand is $\cosh x$ itself. The [3]
catenary is the one curve on this specification where the square root disappears
without a substitution. The length is $[\sinh x]$ from 0 to 1 = **sinh 1**, about 1.175. M1 for using
the identity, M1 for integrating $\cosh x$, A1 for $\sinh 1$.

SECTION B · Standard

- B1** $dx/dt = 2t$ and $dy/dt = 3t^2$, so the integrand is $\sqrt{(4t^2 + 9t^4)} = t\sqrt{4 + 9t^2}$ for $t \geq 0$. [5]
Substituting $u = 4 + 9t^2$ gives $(1/27)(4 + 9t^2)^{3/2}$, which runs from $8/27$ to $13^{3/2}/27$. The
length is $(13\sqrt{13} - 8)/27$, about **1.440**. M1 for the parametric integrand, A1 for $t\sqrt{4 + 9t^2}$,
M1 for the substitution, A1 for the antiderivative, A1 for the length.
- B2** In polar form the integrand is $\sqrt{r^2 + (dr/d\theta)^2}$. Here $dr/d\theta = e^\theta$ as well, so the [4]
bracket is $2e^{2\theta}$ and the integrand is $\sqrt{2} e^\theta$. Integrating gives $\sqrt{2}(e^\pi - 1)$, about **31.31**. M1 for
the polar integrand, A1 for $\sqrt{2} e^\theta$, M1 for integrating, A1 for the length. Every polar arc
length needs r and its derivative, never r alone.
- B3** The surface integral is $2\pi \int y \, ds = 2\pi \int x^3 \sqrt{1 + 9x^4} \, dx$ from 0 to 1. Substituting $u = 1$ [5]
 $+ 9x^4$ gives $(\pi/27)(1 + 9x^4)^{3/2}$, which runs from $\pi/27$ to $10\sqrt{10} \pi/27$. The area is
 $(\pi/27)(10\sqrt{10} - 1)$, about **3.563** square units. M1 for the surface integral, A1 for the
integrand, M1 for the substitution, A1 for the antiderivative, A1 for the area.

SECTION C · Stretch

- C1** Since $ds = \cosh x \, dx$, the integral is $2\pi \int \cosh^2 x \, dx$ from 0 to 1. Writing $\cosh^2 x =$ [5]
 $(\cosh 2x + 1)/2$ gives $2\pi[x/2 + \sinh(2x)/4]$, so the area is $2\pi(1/2 + (\sinh 2)/4) = \pi +$
 $\pi(\sinh 2)/2$, about **8.839** square units. M1 for the surface integral, M1 for the double
angle identity, A1 for the integration, M1 for the limits, A1 for the exact area. Without the
double angle identity there is nothing to integrate.

C2 $dr/d\theta = -2 \sin \theta$, so $r^2 + (dr/d\theta)^2 = 4(1 + \cos \theta)^2 + 4 \sin^2 \theta = 8 + 8 \cos \theta = 16 \cos^2(\theta/2)$. The square root is $4|\cos(\theta/2)|$, and $\cos(\theta/2) \geq 0$ for $0 \leq \theta \leq \pi$, so the upper half has length $\int 4 \cos(\theta/2) d\theta = [8 \sin(\theta/2)] = 8$. By symmetry in the initial line the perimeter is **16**. M1 for $r^2 + (dr/d\theta)^2$, A1 for $16 \cos^2(\theta/2)$, B1 for the modulus and its range of validity, M1 for integrating, A1 for the half length, A1 for the perimeter. Dropping the modulus and running the integral straight from 0 to 2π returns 0, since $\cos(\theta/2)$ is negative on the second half.
