inkMaths

ANSWER BOOK

Areas, parametric curves and the limit of a sum

Integration · Year 12–13

7 questions · 32 marks

NAVY EDITION

SECTION A · Warm-up

A1 Setting $y = 0$ gives $4 - x^2 = 0$, so the right-hand limit is $x = 2$, and the y-axis supplies $x = 0$. Then $\int_0^2 (4 - x^2)\,dx = [4x - x^3/3]$ from 0 to 2 $= 8 - 8/3 = 16/3$. B1 for the upper limit, M1 for the integration, A1 for $16/3$. Reading the upper limit off a rough sketch rather than solving $y = 0$ is the usual route to $x = 4$ and a wrong answer. [3]

A2 Solve $x = x^2$ for the limits, giving $x = 0$ and $x = 1$. Between them the line rides above the parabola, so the area is $\int_0^1 (x - x^2)\,dx = 1/2 - 1/3 = 1/6$. M1 for equating to find the limits, A1 for $x = 0$ and $x = 1$, M1 for integrating the difference, A1 for $1/6$. Top minus bottom, integrated across the enclosed stretch. Subtract the other way round and you get $-1/6$; a negative area earns no accuracy mark, however tidy the integration. [4]

SECTION B · Standard

B1 Equating gives $9 - x^2 = x^2 + 1$, so $x^2 = 4$ and the curves cross at $x = \pm 2$. The vertical gap is $(9 - x^2) - (x^2 + 1) = 8 - 2x^2$, and $\int_{-2}^2 (8 - 2x^2)\,dx = [8x - 2x^3/3]$ from -2 to 2 $= 32/3 - (-32/3) = 64/3$. M1 A1 for the limits $x = \pm 2$, M1 for the difference of the curves, M1 for the integration, A1 for $64/3$. One integral of the height difference handles both boundaries at once. Integrating each curve separately and subtracting works too, but it doubles the chances of a slip at the negative limit. [5]

B2 $x = 8$ gives $t = 2$, and $dx/dt = 3t^2$, so the area is $\int_0^2 y\,(dx/dt)\,dt = \int_0^2 3t^2 \times 3t^2\,dt = \int_0^2 9t^4\,dt = [9t^5/5]$ from 0 to 2 $= 288/5$. For the check, $t = x^{1/3}$ gives $y = 3x^{2/3}$, and $\int_0^8 3x^{2/3}\,dx = [(9/5)x^{5/3}]$ from 0 to 8 $= 288/5$. B1 for $t = 2$, M1 A1 for the integral of y times dx/dt, A1 for $288/5$, M1 A1 for the Cartesian check. The mark most often dropped is the change of limits. $x = 8$ has to become $t = 2$ before the integral in t is written down, and candidates who carry 0 and 8 into a t-integral lose everything after the first line. [6]

B3 The limit is $\int_1^3 x^2\,dx$, and that evaluates to $[x^3/3]$ from 1 to 3 $= 9 - 1/3 = 26/3$. B1 for the integral with the right limits, M1 for integrating, A1 for $26/3$. Each term $f(x)\,\delta x$ is one strip's area, Σ adds the strips, and the limit turns the sum into the stretched S with dx sitting where δx used to. [3]

SECTION C · Stretch

C1 $dx/dt = -3 \sin t$, and as t runs from 0 to $\pi/2$ the point travels leftwards from (3, 0) to (0, 1). Sweeping the other way, the area is $\int_{\pi/2}^{0} y\,(dx/dt)\,dt = 3\int_{0}^{\pi/2} \sin^2 t\,dt$. Now use $\sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t$, giving $3[t/2 - (\sin 2t)/4]$ from 0 to $\pi/2 = 3(\pi/4 - 0) = 3\pi/4$ as required. M1 for the integral of y times dx/dt, A1 for handling the reversed limits, M1 A1 for the identity and the integration, A1 for the printed result. The identity is unavoidable, since $\sin^2 t$ cannot be integrated as it stands. Leave a stray minus sign there and the printed answer can only be reached by a fudge, which shows in the working. [5]

C2 $dx/dt = 2 \cos t$, and x increases from 0 to 2 as t runs from 0 to $\pi/2$, so the area is $\int_{0}^{\pi/2} 3 \sin 2t \times 2 \cos t\,dt$. Replace $\sin 2t$ by $2 \sin t \cos t$ and the integrand becomes $12 \sin t \cos^2 t$. That is a reverse chain, since the derivative of $\cos^3 t$ is $-3 \sin t \cos^2 t$, so the antiderivative is $-4 \cos^3 t$. Evaluating from 0 to $\pi/2$ gives $0 - (-4) = 4$. M1 for dx/dt, M1 A1 for the integral in t with the correct limits, M1 for replacing sin 2t by 2 sin t cos t, A1 for the antiderivative, A1 for the printed result. Most attempts stall trying to integrate $6 \sin 2t \cos t$ as it stands. The double angle has to be unpicked before any integration can start. [6]

