inkMaths

ANSWER BOOK

Calculus with hyperbolic functions

Hyperbolic functions · Year FM

7 questions · 27 marks

NAVY EDITION

SECTION A · Warm-up

A1 $\sinh' = \cosh$ and $\cosh' = \sinh$. The pair swap with both signs positive. B1 B1 for the two derivatives. Unlike cos, whose derivative is $-\sin$, cosh differentiates with no minus anywhere. [2]

A2 Chain rule gives $dy/dx = 3 \sinh 3x$. At $x = 0$, $\sinh 0 = 0$, so the gradient is 0. M1 A1 for the derivative, A1 for the gradient. The curve has its minimum there, of value $\cosh 0 = 1$. [3]

SECTION B · Standard

B1 Gradient $\sinh(\ln 2) = 3/4$ and value $\cosh(\ln 2) = 5/4$, so $y = 5/4 + (3/4)(x - \ln 2)$. M1 A1 for the gradient, B1 for the value of y, A1 for the tangent. Exact hyperbolic values at logarithmic points come straight from the definitions, with no calculator required. [4]

B2 $\int \sinh = \cosh$, both signs positive, so the value is $[\cosh x]$ from 0 to $\ln 3 = \cosh(\ln 3) - 1 = 5/3 - 1 = 2/3$. The value $\cosh(\ln 3) = 5/3$ comes from $(3 + 1/3)/2$. M1 for the antiderivative, M1 for $\cosh(\ln 3) = 5/3$, A1 for 2/3. Marks are lost here by importing the minus sign from $\int \sin x \, dx = -\cos x$. [3]

B3 With $a = 3$ the integral is $\operatorname{arcosh}(x/3)$ from 3 to 5, that is $\operatorname{arcosh}(5/3) - \operatorname{arcosh} 1$. In log form this is $\ln(5/3 + \sqrt{(25/9 - 1)}) = \ln(5/3 + 4/3) = \ln 3$, and $\operatorname{arcosh} 1 = 0$. The answer is $\ln 3$. M1 A1 for the arcosh form, M1 for the logarithmic form, A1 for $\ln 3$. [4]

B4 $\cosh^2 x = (1 + \cosh 2x)/2$, so the integral is $[x/2 + \sinh 2x/4]$ from 0 to $1 = 1/2 + \sinh 2/4$. Since $\sinh 2 = 2 \sinh 1 \cosh 1$, this is $(1 + \sinh 1 \cosh 1)/2 \approx 1.407$. M1 for rearranging the identity, M1 for integrating, A1 for the antiderivative, A1 for the exact value. The double angle identity carries no sign flip, unlike its cos 2x counterpart. [4]

SECTION C · Stretch

C1 $dx = 2 \cosh u \, du$ and $x^2 + 4 = 4 \sinh^2 u + 4 = 4 \cosh^2 u$, so $\sqrt{(x^2 + 4)} = 2 \cosh u$ and the integral becomes $\int 4 \cosh^2 u \, du$. The limits transform too, with $x = 0$ giving $u = 0$ and $x = 2\sqrt{3}$ giving $\sinh u = \sqrt{3}$, so $u = \operatorname{arsinh} \sqrt{3} = \ln(\sqrt{3} + 2)$. Using $\cosh^2 u = (1 + \cosh 2u)/2$ turns the integral into $2[u + \sinh u \cosh u]$. At the upper limit $\sinh u = \sqrt{3}$ and $\cosh u = 2$, giving $2[\ln(2 + \sqrt{3}) + 2\sqrt{3}] = \mathbf{2} \ln(2 + \sqrt{3}) + 4\sqrt{3}$, about 9.562. M1 for $dx = 2 \cosh u \, du$, M1 A1 for reducing the root to $2 \cosh u$, B1 for the transformed limits, M1 for the double angle identity, A1 for the antiderivative, A1 for the printed result. The two traps are leaving the limits in terms of x and forgetting that cosh u is the positive root throughout. [7]