inkMaths

ANSWER BOOK

Diagonalisation and the Cayley-Hamilton theorem

Further Pure 2 · Year FM

6 questions · 23 marks

NAVY EDITION

SECTION A · Warm-up

A1 Put the eigenvectors in as the columns of P, so P has rows **(1,** 1) and **(1,** -2). [2]
Then D is diagonal with the matching eigenvalues in the same order, **5** then 2. B1 for P, B1 for D in the matching order. Swapping the columns of P works provided the entries of D are swapped with them. Mismatching the order is the standard error, and it costs both marks.

A2 Every square matrix satisfies its own characteristic equation. Here $\det(M - \lambda I) = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10$, so the claim is $M^2 - 7M + 10I = 0$. [3]
M^2 has rows (18, 7) and (14, 11), 7M has rows (28, 7) and (14, 21), and 10I has 10 on the diagonal. Then $M^2 - 7M + 10I$ has entries $18 - 28 + 10$, $7 - 7$, $14 - 14$ and $11 - 21 + 10$, all zero, **as** required. B1 for the statement of the theorem, M1 for forming $M^2 - 7M + 10I$, A1 for the zero matrix. Verifying means reaching the zero matrix; quoting the characteristic equation alone is one mark.

SECTION B · Standard

B1 Take P with rows (1, 1) and (1, -2) and D diagonal with 5 and 2, so that $M = PDP^{-1}$ and $M^5 = PD^5P^{-1}$. D^5 is diagonal with entries **3125** and **32**; only the diagonal entries are ever raised to a power. [5]
Since $\det P = -3$, P^{-1} is (-1/3) times the matrix with rows (-2, -1) and (-1, 1).
Multiplying the three matrices in the order P, then D^5, then P^{-1} gives M^5 with rows **(2094,** 1031) and **(2062,** 1063). M1 for using M raised to a power as P times D to that power times P^{-1}, A1 for 3125 and 32, M1 for P^{-1}, M1 for the triple product, A1 for the four entries. Reversing P and P^{-1} gives a different matrix and loses the accuracy marks, since matrix multiplication does not commute.

B2 The characteristic equation is $\lambda^2 - 7\lambda + 10 = 0$, so $M^2 - 7M + 10I = 0$. Multiply throughout by M^{-1}: $M - 7I + 10M^{-1} = 0$, so $M^{-1} = (7I - M)/10$. That is one tenth of the matrix with rows (3, -1) and (-2, 4). M1 for multiplying the relation through by M^{-1}, A1 for $M^{-1} = (7I - M)/10$, M1 for evaluating $7I - M$, A1 for the entries. Checking, M times that matrix is 10I, so dividing by 10 gives the identity. The method needs no cofactors. [4]

B3 Take the matrix with rows (1, 1) and (0, 1). Its characteristic equation is $(1 - \lambda)^2 = 0$, so 1 is a repeated eigenvalue, but solving $(M - I)v = 0$ gives only multiples of (1, 0). [3]
With one independent eigenvector there is no invertible P of eigenvectors, so no diagonalisation exists. B1 for a suitable matrix, M1 for solving for the eigenvectors, A1 for only one independent eigenvector, so no invertible P. A repeated eigenvalue is a warning rather than a verdict, since some repeated cases still supply two independent eigenvectors.

SECTION C · Stretch

C1 Cayley-Hamilton gives $M^2 = 7M - 10I$, so every power collapses onto M and I, and $M^n = a_nM + b_nI$ for some numbers a_n and b_n. [6]

To find them, use the eigenvalues. If $Mv = \lambda v$ then $M^nv = \lambda^nv$, and applying the relation to that same eigenvector gives $\lambda^n = a_n\lambda + b_n$. Both eigenvalues obey it, so $5a_n + b_n = 5^n$ and $2a_n + b_n = 2^n$.

Subtracting, $3a_n = 5^n - 2^n$, which is the given a_n, and back-substituting gives $\mathbf{b_n} = (5\cdot2^n - 2\cdot5^n)/3$. Setting up the pair of equations from the two eigenvalues is where the method marks sit; grinding out M^3, M^4, M^5 one at a time reaches the same place far more slowly.

At $n = 6$: $a = (15625 - 64)/3 = 5187$ and $b = (320 - 31250)/3 = -10310$, so $M^6 = 5187M - 10310I$, with rows **(10438,** 5187) and **(10374,** 5251). B1 for $M^2 = 7M - 10I$, M1 for applying the relation to an eigenvector, A1 for the pair of simultaneous equations, A1 for b_n, M1 for substituting $n = 6$, A1 for the entries of M^6.

A check at $n = 2$ gives $a = 7$ and $b = -10$, which is the Cayley-Hamilton relation itself.