inkMaths

ANSWER BOOK

Estimators, standard error and confidence intervals

Further Statistics 2 · Year FM

6 questions · 22 marks

NAVY EDITION

SECTION A · Warm-up

A1 An estimator is unbiased if its expected value equals the parameter it estimates, so that it is right on average over repeated samples. The sample variance with divisor $n - 1$, $s^2 = \Sigma(x - \text{sample mean})^2/(n - 1)$, is unbiased for σ^2; dividing by n instead gives an estimate that is too small on average. B1 for the definition, B1 for naming s^2 with divisor $n - 1$. [2]

A2 Standard error = $12/\sqrt{36}$ = **2**. The interval is $48 \pm 1.96 \times 2 = 48 \pm 3.92$, that is **(44.08,** 51.92). B1 for the standard error, M1 for $48 \pm 1.96 \times 2$, A1 for the interval. [3]

SECTION B · Standard

B1 Standard error = $12/\sqrt{36} = 2$, unchanged, since the confidence level affects only the multiplier. [4]
Using $z = 2.5758$: $48 \pm 2.5758 \times 2 = 48 \pm 5.15$, that is **(42.85,** 53.15).
It is wider than the 95% interval by about 1.23 units at each end. More confidence is bought with less precision, and only a larger sample improves both at once. B1 for $z = 2.5758$, M1 for $48 \pm z \times 2$, A1 for the interval, B1 for the comparison. Quoting 2.58 rather than 2.5758 is acceptable, but 1.96 for a 99% interval loses every accuracy mark that follows.

B2 The total width is $2 \times 1.96 \times 12/\sqrt{n}$, and the requirement is that this is at most 4. [4]
Working with the total width rather than the half-width is where most answers go wrong, and it changes the answer by a factor of four.
Rearranging: $\sqrt{n} \geq 2 \times 1.96 \times 12/4 = 11.76$, so $n \geq 138.3$.
Since n must be a whole number and the inequality runs upwards, round **up** to **n** = 139.
Rounding down to 138 gives an interval slightly too wide and loses the final mark. M1 for an inequality in the total width, A1 for $\sqrt{n} \geq 11.76$, dM1 for reaching $n \geq 138.3$, A1 for 139.
Halving the width again would need four times as many readings, since n appears under a square root.

B3 The population mean is a fixed number rather than a random one, so once the interval has been calculated it either contains μ or it does not, and no probability remains. What varies from sample to sample is the interval, not μ. [3]
The correct statement is that the method produces intervals of which 95% would contain μ in repeated sampling. The confidence attaches to the procedure, not to the one interval in front of you. B1 for μ being a fixed number, B1 for the interval being what varies, B1 for the repeated-sampling statement.

SECTION C · Stretch

C1 **Unbiasedness.** $E(\bar{X}) = E(\bar{Y}) = \mu$, so $E(T) = \lambda\mu + (1 - \lambda)\mu = \mu$ for every λ. The coefficients were chosen to add to 1, which is exactly what unbiasedness requires, so no value of λ can be ruled out on those grounds. Unbiasedness alone never settles which estimator to use. [6]

Variance. The two sample means are independent, with $\mathrm{Var}(\bar{X}) = \sigma^2/4$ and $\mathrm{Var}(\bar{Y}) = \sigma^2/6$. Each coefficient squares:
$\mathrm{Var}(T) = \lambda^2\sigma^2/4 + (1 - \lambda)^2\sigma^2/6$.

Minimising. Differentiate with respect to λ and set the result to zero: $\lambda\sigma^2/2 - (1 - \lambda)\sigma^2/3 = 0$. The σ^2 cancels, which is the sign that the answer cannot depend on it.
So $3\lambda = 2(1 - \lambda)$, giving $5\lambda = 2$ and $\boldsymbol{\lambda} = 0.4$. The second derivative is positive, so this is a minimum; say so, because it is a mark.
Then $\mathrm{Var}(T) = 0.16\sigma^2/4 + 0.36\sigma^2/6 = 0.04\sigma^2 + 0.06\sigma^2 = \mathbf{0.1\boldsymbol{\sigma}^2}$, that is $\sigma^2/10$.
B1 for $E(T) = \mu$ for every λ, M1 A1 for $\mathrm{Var}(T) = \lambda^2\sigma^2/4 + (1 - \lambda)^2\sigma^2/6$, M1 for differentiating and equating to zero, A1 for $\lambda = 0.4$, A1 for $0.1\sigma^2$.
Note what that means. $\sigma^2/10$ is the variance of the mean of all ten observations, so the best weighting is the one that treats every original reading equally. Weighting the two sample means equally, at $\lambda = 0.5$, gives $5\sigma^2/48 \approx 0.104\sigma^2$, which is worse.