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ANSWER BOOK

# Graphs: order, Eulerian paths and planarity

Decision Mathematics 1 · Year FM

7 questions · 24 marks

NAVY EDITION


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## SECTION A · Warm-up

**A1** Every edge contributes one to the order of each of its two ends, so the orders of all the nodes total twice the number of edges, which is even. The even-order nodes contribute an even amount, so the odd-order ones must too, and that is only possible if there is an even number of them. B1 for the orders totalling twice the edges, B1 for the deduction. [2]

**A2** The orders total 18, which is twice the nine edges, as required. Four nodes are odd, and four is more than two, so the graph is **neither** Eulerian nor semi-Eulerian: no trail can use every edge exactly once. B1 for the total of 18, B1 for four odd nodes, B1 for the classification. [3]

## SECTION B · Standard

**B1** $K_n$ has n(n - 1)/2 edges, so K5 has **10** and K6 has **15**. In $K_n$ every node has order n - 1. For K5 that is 4, even, so **K5** is Eulerian. For K6 it is 5, odd, and all six nodes are odd, so K6 is neither Eulerian nor semi-Eulerian. B1 B1 for the two edge counts, M1 for the order n - 1, A1 for the classification of each. [4]

**B2** It needs a Hamiltonian cycle, one visiting every node exactly once. Redraw that cycle as a circle with the remaining edges as chords, then take the chords in turn, placing each inside or outside the circle so that it crosses nothing already placed. If every chord can be placed the graph is planar and the drawing proves it. If some chord conflicts with what is inside and also with what is outside, no placement exists and the graph is **not** planar. B1 for the Hamiltonian cycle, B1 for redrawing it as a circle with chords, B1 for placing each chord inside or outside, B1 for the conflicting chord. [4]

**B3** Isomorphism requires a one-to-one correspondence of nodes that preserves every edge, and matching totals do not supply one. The orders could be arranged differently: in one graph the two order-4 nodes might be joined and in the other not. To prove isomorphism you must give the correspondence explicitly and check every edge. B1 for matching totals not supplying a correspondence, B1 for what a proof requires. [2]

**B4** Two odd nodes makes the network **semi-Eulerian**. A trail using every path exactly once exists, but it must **start** at one odd junction and finish at the other, so the warden cannot end where they began. Building a path between the two odd junctions raises both orders by one. Every node is then even, the network is **Eulerian**, and a closed round exists: one extra path walked, but a return to the start. B1 for semi-Eulerian, B1 for the start and finish at the odd junctions, B1 for the network becoming Eulerian. [3]

## SECTION C · Stretch

**C1** K5 has 5 × 4/2 = 10 edges, and the cycle ABCDEA uses five of them. Redraw that cycle as a circle. The five edges left over become the chords **AC,** AD, BD, BE and CE. Two chords clash when their ends interleave round the circle, so that drawing both on the same side would force a crossing. Chords sharing an end never clash. Working through the pairs: AC clashes with BD and BE; AD clashes with BE and CE; BD clashes with CE. That is **five** clashes, and every chord is in exactly two of them. [6]

Follow them round: AC clashes with BD, BD with CE, CE with AD, AD with BE, and BE back with AC. The clashes form a **single** cycle of length five.

Placing the chords is the same as colouring that cycle with two colours, inside and outside, so that no clashing pair shares a colour. Go round the five-cycle alternating: the colours must alternate at every step, but after an odd number of steps you arrive back at the start needing the opposite colour to the one it already has. No such split exists, so whichever way the first four chords are placed, the fifth clashes with something inside and with something outside. K5 is therefore **not** planar.

The argument needs the odd length. A clash cycle of even length alternates back to itself consistently and the graph would be planar, so quoting 'the chords clash' without counting them proves nothing.

B1 for the five chords, M1 A1 for the clashes, M1 for the clash cycle of length five, M1 A1 for the two-colouring argument.