inkMaths

ANSWER BOOK

Groups and their axioms

Further Pure 2 · Year FM

7 questions · 28 marks

NAVY EDITION

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SECTION A · Warm-up

A1 Closure: the operation on two elements of the set gives an element of the set. Associativity: $(ab)c = a(bc)$ for all elements. Identity: there is an element e with $ea = ae = a$ for every a. Inverses: every a has an element a^{-1} in the set with $aa^{-1} = a^{-1}a = e$. B1 for closure and associativity, B1 for the identity, B1 for inverses. Commutativity is not required. [3]

A2 The products are $3 \times 5 = 15 \equiv 7$, $3 \times 7 = 21 \equiv 5$ and $5 \times 7 = 35 \equiv 3$, and each element squares to 1, so the set is closed. Multiplication modulo 8 is associative, 1 is the identity, and every element is its own inverse. The identity has order 1; 3, 5 and 7 each have order 2. M1 for the closure check, A1 for the remaining axioms, B1 B1 for the orders. [4]

SECTION B · Standard

B1 Rows for 1, 2, 3, 4 read (1, 2, 3, 4), (2, 4, 1, 3), (3, 1, 4, 2) and (4, 3, 2, 1). Every entry lies in the set, the identity 1 appears exactly once in each row and column so inverses exist, and modular multiplication is associative. Powers of 2 give 2, 4, 3, 1, so 2 has order 4 and generates the whole group, so it is cyclic. (3 works as well; 4 has order 2 and does not.) M1 A1 for the table, M1 for the group check, A1 for the conclusion, B1 for the generator. [5]

B2 The order of a group is the number of elements it contains; the order of an element is the least positive power of it that gives the identity. The symmetry group of an equilateral triangle has order 6, three rotations and three reflections. The rotation through 120° has order 3, since three of them return the triangle to its starting position, while each reflection has order 2. B1 for the order of a group, B1 for the order of an element, B1 for the triangle example. [3]

B3 Multiplying the matrix for m by the matrix for n gives rows $(1, m + n)$ and $(0, 1)$, so the set is closed and the operation behaves like adding the integers. Matrix multiplication is associative, $n = 0$ gives the identity matrix, and the matrix for $-n$ inverts the matrix for n. Since the operation is addition of the n values, the group is isomorphic to the integers under addition. M1 A1 for the product and closure, B1 for the identity and inverses, B1 for the isomorphism. [4]

B4 With 0 included there is no inverse for 0, since no residue multiplied by 0 gives the identity 1. Removing it leaves $\{1, 2, 3\}$, but $2 \times 2 = 4 \equiv 0$, which is outside the set, so closure fails and 2 has no inverse either. The multiplicative residues modulo n form a group only when the residues kept are those coprime to n, which for 4 means $\{1, 3\}$. B1 for 0 having no inverse, M1 for testing closure on the remaining set, A1 for $2 \times 2 \equiv 0$. [3]

SECTION C · Stretch

C1 Every element is its own inverse, since $x^2 = e$ gives $x^{-1} = x$. Take any a and b in G. Then ab lies in G, so $(ab)^2 = e$ and ab is its own inverse, that is $ab = (ab)^{-1}$. But $(ab)^{-1} = b^{-1}a^{-1} = ba$, using the order reversal for the inverse of a product and then the fact that each element is self-inverse. Hence $ab = ba$ for all a and b, so G is abelian. $\{1, 3, 5, 7\}$ under multiplication modulo 8 has order 4 and every element squares to 1, so it is an example. The whole proof turns on applying the hypothesis to the product ab rather than to a and b separately. B1 for every element being self-inverse, M1 for applying the hypothesis to ab, A1 for the reversal of the inverse, A1 for $ab = ba$, A1 for the conclusion, B1 for the example. [6]