inkMaths

ANSWER BOOK

Mean values and improper integrals

Further calculus · Year FM

7 questions · 26 marks

NAVY EDITION

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SECTION A · Warm-up

A1 The integral of f from a to b divided by b - a. It is the height of the rectangle on the same base that has the same area as the region under the curve, the level the curve would settle to if its area were poured flat. B1 for the definition, B1 for the geometric description. [2]

A2 $\int \sin x\, dx$ from 0 to π = [-cos x] = 2. Dividing by the width π gives mean 2/π ≈ 0.637. M1 for integrating, A1 for 2, A1 for 2/π. Less than the maximum 1, as the curve spends its time rising to the peak and falling away. [3]

SECTION B · Standard

B1 $\int e^{2x}\, dx$ from 0 to 1 = $[e^{2x}/2]$ = $(e^2 - 1)/2$. The width is 1, so the mean is $(e^2 - 1)/2$ ≈ 3.19. M1 for integrating, A1 for the antiderivative, M1 for dividing by the width, A1 for the exact mean. The mean sits well below the endpoint value e^2 ≈ 7.39 because an exponential spends most of any interval far below its final value. [4]

B2 Integrate to a limit t: $[-2x^{-1/2}]$ from 1 to t = 2 - 2/√t. As t → ∞ the second term vanishes, so the integral converges to 2. M1 for integrating to a limit t, A1 for 2 - 2/√t, M1 for letting t → ∞, A1 for 2. The power 3/2 is above the boundary case 1, which is what buys convergence. [4]

B3 The integrand is unbounded at 0, so integrate from t to 4: [2√x] = 4 - 2√t. As t → 0^+ this tends to 4, so the integral converges to 4. M1 for replacing the lower limit by t, A1 for 4 - 2√t, M1 for the limit as t → 0, A1 for 4. The spike at zero is infinitely tall but narrow enough to hold finite area. [4]

B4 The $1/x^{3/2}$ integral converges, to 2. The 1/x integral gives ln t, which grows without bound. B1 for naming the convergent integral, B1 for the divergence of ln t, B1 for the explanation in terms of rate. Tending to zero is not enough on its own, since what settles the question is how fast. Powers of x below -1 shrink quickly enough to trap finite area, and 1/x sits exactly on the divergent side of the boundary. [3]

SECTION C · Stretch

C1 $\ln x$ is unbounded below as $x \to 0^+$, so the integral is improper at the lower limit and must be handled as a limit. Integrating by parts with $u = \ln x$ and $dv = dx$ gives $\int \ln x \, dx = x \ln x - x$. Between t and 1 this is $(0 - 1) - (t \ln t - t) = -1 - t \ln t + t$. As $t \to 0^+$ the term $t \ln t$ tends to 0 by the given result and t tends to 0, so the limit exists and the integral converges to **−1**. B1 for recognising the integral is improper at 0, M1 for integration by parts, A1 for $x \ln x - x$, M1 for evaluating between t and 1, A1 for $-1 - t \ln t + t$, A1 for the value -1. The value is negative because $\ln x$ lies below the axis throughout $0 < x < 1$. Writing $[x \ln x - x]$ from 0 to 1 straight out is worth no marks, since $x \ln x$ is undefined at 0 and the limit is exactly what needs showing. [6]