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ANSWER BOOK

Newton-Raphson and the trapezium rule

Numerical methods · Year 13

7 questions · 23 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $x_{n+1} = x_n - f(x_n)/f'(x_n)$. B1 for the formula, B1 for the tangent interpretation. [2]
Geometrically it slides down the tangent at the current point to where that tangent crosses the axis, then starts again from there. The minus sign is part of the formula, not an optional extra.

A2 Area $\approx (h/2)[y_0 + y_n + 2(y_1 + \ldots + y_{n-1})]$. B1 for the h/2 bracket with the end ordinates, B1 for doubling those between. The two end ordinates count once and everything between counts twice, because each interior ordinate is a side of two neighbouring trapezia. With n strips there are n + 1 ordinates, and miscounting them is the most reliable way to lose the method mark. [2]

SECTION B · Standard

B1 $f(1.5) = -0.125$ and $f'(1.5) = 3(1.5)^2 + 1 = 7.75$, so $x_1 = 1.5 + 0.125/7.75 = 1.516129\ldots$. [4]
Then $f(x_1) = 0.00117\ldots$, $f'(x_1) = 7.8959\ldots$, and $x_2 = 1.51598$ to 5 decimal places. M1 for f and f′ at 1.5, A1 for the first iterate, M1 for the second iteration, A1 for 1.51598. The root is 1.5159802..., so x_1 was right to 3 decimal places and x_2 is right to 7. Newton-Raphson roughly doubles the number of correct digits at each step when it is behaving. Keep the unrounded x_1 in the calculator for the second step, because rounding it to 1.5161 first corrupts the fifth decimal place of x_2.

B2 With 4 strips over [1, 2], h = 0.25 and the five ordinates are ln 1 = 0, ln 1.25 = 0.2231, ln 1.5 = 0.4055, ln 1.75 = 0.5596 and ln 2 = 0.6931. So the estimate is $(0.25/2)[0 + 0.6931 + 2(0.2231 + 0.4055 + 0.5596)] = 0.125 \times 3.0696 = 0.3837$. B1 for h = 0.25, M1 for the five ordinates, M1 for a correct trapezium rule structure, A1 for 0.3837. Work to more figures than the answer requires inside the bracket, since rounding each ordinate to 2 decimal places first is enough to shift the fourth decimal place of the result. [4]

B3 ln x is concave, so the curve bends above every chord drawn across it. Each trapezium's sloped top therefore runs below the curve and each strip undershoots the true area. For a convex curve the same reasoning flips and the rule overestimates. B1 for ln x being concave, B1 for each chord lying below the curve. A sketch showing one chord below the curve is worth more than any assertion about the sign of the error. [2]

SECTION C · Stretch

C1 $f'(x) = 3x^2 - 3$, so $f'(1) = 0$ while $f(1) = -1$. The iteration divides by $f'(x_n)$, so the first step is undefined. Geometrically the tangent at $x = 1$ is horizontal and never meets the x-axis. Taking $x_0 = 0.9$ gives $f(0.9) = -0.971$ and $f'(0.9) = -0.57$, so $x_1 = 0.9 - (-0.971)/(-0.57) = -0.804$. The starting value was close to the root 0.347, yet the near-horizontal tangent has flung the iterate past it to the far side. Continuing the iteration from there does converge, but to 1.532, a different root altogether. B1 for $f'(1) = 0$, B1 for the division being undefined, B1 for the horizontal tangent, M1 for the iteration from 0.9, A1 for the value of x_1, B1 for the comment on the root reached. The lesson is that Newton-Raphson gives no warning when it misbehaves. A sketch, or a sign-change check on the final answer, is what catches it. [6]

C2 Double the number of strips. Narrower trapezia hug the curve more closely, so 8 strips give 0.3856 against 0.3837 from 4. The check is direction and size together. B1 for doubling the strips, B1 for the estimate staying below the exact value, B1 for each estimate being closer than the last. Because ln x stays concave across the whole interval, every estimate must lie below the exact 0.3863 and each one must be closer than the last. An estimate that jumps above 0.3863, or moves away from it, points to an arithmetic error rather than to the method. [3]