inkMaths

ANSWER BOOK

Proof by induction: sums and series

Further proof · Year FM

7 questions · 26 marks

NAVY EDITION

SECTION A · Warm-up

A1 A base case (the statement holds at $n = 1$) and an inductive step (truth at $n = k$ forces truth at $n = k + 1$). The base case starts the chain and the step passes truth along it, so every positive integer is eventually reached. B1 B1 for the base case and the inductive step. [2]

A2 At $n = 1$ the left side is 1 and the right side is $1^2 = 1$, so the base case holds. Hypothesis: assume $1 + 3 + \dots + (2k - 1) = k^2$ for some positive integer k. B1 for the base case, B1 for the hypothesis. [2]

SECTION B · Standard

B1 Assuming the sum to k terms is k^2, adding the next odd number gives $k^2 + (2k + 1) = (k + 1)^2$. That is the claim at $n = k + 1$. With the base case established, the statement holds for all positive integers n by induction. M1 for adding the next odd number, A1 for $(k + 1)^2$, A1 for the conclusion. [3]

B2 Base: $n = 1$ gives $1 = 1 \times 2/2$. Step: assume the sum to k is $k(k + 1)/2$; adding $k + 1$ gives $k(k + 1)/2 + (k + 1) = (k + 1)(k + 2)/2$, the formula at $k + 1$. Both parts hold, so the result is true for all n. B1 for the base case, M1 for adding $k + 1$ to the assumed sum, dM1 for taking out the common factor, A1 for $(k + 1)(k + 2)/2$, A1 for the conclusion. The common factor $(k + 1)$ does the algebra, so take it out before expanding anything. [5]

B3 Base: $n = 1$ gives $2 = 2^2 - 2 = 2$. Step: assume the sum to k is $2^{k+1} - 2$. Adding the next term 2^{k+1} gives $2 \times 2^{k+1} - 2 = 2^{k+2} - 2$, which is the claim at $k + 1$. A spot check at $n = 4$: B1 for the base case, B1 for the hypothesis, M1 for adding 2^{k+1}, A1 for $2^{k+2} - 2$, A1 for the conclusion. $2 + 4 + 8 + 16 = 30 = 2^5 - 2$. [5]

B4 The step only passes truth along; without a first true case there is nothing to pass. The false formula $n^2 + 1$ also survives the inductive step, since adding $2k + 1$ to $k^2 + 1$ gives $(k + 1)^2 + 1$. It fails only at the base case, $1 \neq 2$, which is exactly the part the student skipped. B1 for the step needing a first true case, M1 for the false formula surviving the step, A1 for its failure at $n = 1$. [3]

SECTION C · Stretch

C1 Base: at $n = 1$ the sum is $1 \times 2 = 2$, and the formula gives $0 \times 4 + 2 = 2$, so the two agree. Step: assume the sum to k terms is $(k - 1)2^{k+1} + 2$. Adding the next term $(k + 1)2^{k+1}$ gives $(k - 1)2^{k+1} + (k + 1)2^{k+1} + 2$. Take out the common factor 2^{k+1}, leaving $2^{k+1}(k - 1 + k + 1) + 2 = 2^{k+1}(2k) + 2 = k2^{k+2} + 2$. The formula at $n = k + 1$ reads $((k + 1) - 1)2^{k+2} + 2 = k2^{k+2} + 2$, which matches. True at $n = 1$ and inherited at every step, so true for all positive integers n by induction. B1 for the base case, M1 for adding $(k + 1)2^{k+1}$, dM1 for taking out 2^{k+1}, A1 for $k2^{k+2} + 2$, A1 for matching the formula at $k + 1$, A1 for the conclusion. Factorising 2^{k+1} out before combining is what keeps the algebra to one line; multiplying everything out in powers of 2 rarely closes. [6]