inkMaths

ANSWER BOOK

Rates of change and building differential equations

Differentiation · Year 12–13

8 questions · 29 marks

NAVY EDITION

SECTION A · Warm-up

A1 $dV/dt = dV/dr \times dr/dt$. B1 for the chain, B1 for the comment. Questions usually give one time rate, such as how fast volume is being pumped in, and ask for the other. The middle factor dV/dr is the one you generate yourself, by differentiating whatever formula links the two quantities. [2]

A2 $A = \pi r^2$, so $dA/dr = 2\pi r$. The chain rule gives $dA/dt = dA/dr \times dr/dt$, that is $10 = 2\pi \times 4 \times dr/dt$, so $dr/dt = 10/(8\pi) = 5/(4\pi) = 0.398$ m per hour. M1 for dA/dr, M1 for the chain, A1 for the value. The same amount of area arrives every hour but it has to spread around an ever longer rim, so the radius grows more and more slowly. [3]

SECTION B · Standard

B1 $V = (4/3)\pi r^3$, so $dV/dr = 4\pi r^2$, which is 64π at $r = 4$. Then $120 = 64\pi \times dr/dt$, so $dr/dt = 120/(64\pi) = 15/(8\pi) = 0.597$ cm per second. M1 for dV/dr, A1 for 64π, M1 for the chain, A1 for 0.597. Differentiate the linking formula with respect to r, not t. The chain supplies the time part, and trying to differentiate V with respect to t directly leaves an r that nothing can be done with. [4]

B2 With edge x, $V = x^3$ gives $dV/dx = 3x^2 = 27$ at $x = 3$, so $27 = 27 \times dx/dt$ and $dx/dt = 1$ cm per second. For the surface, $S = 6x^2$ gives $dS/dx = 12x = 36$ at $x = 3$, so $dS/dt = 36 \times 1 = 36$ cm^2 per second. M1 A1 for the edge rate, M1 A1 for the surface rate. One edge rate feeds both answers, because every rate in the cube has to travel through dx/dt. [4]

B3 $dP/dt = kP$, where k is a positive constant. B1 for the equation, B1 for stating $k > 0$. The sentence names the rate, names what it is proportional to, and the constant absorbs the proportionality. Fixing the sign of k is part of the model rather than an afterthought, since $k < 0$ would describe decay. [2]

B4 The rate of decrease of T is $-dT/dt$ and the excess over the room is $T - 20$, so $-dT/dt = k(T - 20)$, that is $dT/dt = -k(T - 20)$, where k is a positive constant. M1 for identifying $T - 20$, M1 for the minus sign, A1 for the equation, B1 for defining k. The minus sign and the positive k work together to keep the drink cooling while it is hotter than the room, and the model stalls correctly at $T = 20$ because the bracket vanishes there. [4]

SECTION C · Stretch

C1 Let the depth be h and the surface radius r. The semi-vertical angle gives $r = h \tan 30° = h/\sqrt{3}$. The volume of water is $V = (1/3)\pi r^2 h = (1/3)\pi(h^2/3)h = \pi h^3/9$. Differentiating, $dV/dh = \pi h^2/3$, which is 12π at $h = 6$. The chain rule gives $8 = 12\pi \times dh/dt$, so $dh/dt = 8/(12\pi) = 2/(3\pi) = 0.212$ cm s^{-1}. M1 for relating r and h, A1 for V in terms of h alone, M1 for differentiating, A1 for 12π, M1 for the chain, A1 for 0.212. Everything hangs on the first step. Leaving both r and h in the volume formula makes dV/dh meaningless, and the fixed angle is what lets one variable be eliminated. [6]

C2 The net rate of change is the inflow minus the outflow, so $dV/dt = 30 - kV$ with k a positive constant. The volume settles when it stops changing, so $dV/dt = 0$ at $V = 600$, giving $30 = 600k$ and $k = 0.05$ per minute. M1 for the structure inflow minus outflow, A1 for the equation, M1 for setting $dV/dt = 0$, A1 for $k = 0.05$. The settled value is a steady state rather than a limit you have to solve the equation to find, and spotting that saves all the work of separating variables. [4]