inkMaths

ANSWER BOOK

Roots of polynomials

Further algebra and series · Year FM

7 questions · 30 marks

NAVY EDITION

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SECTION A · Warm-up

A1 Divide by 2 first: $z^3 + 3z^2 - 2z + 5 = 0$. Then $\alpha + \beta + \gamma = -3$ and $\alpha\beta\gamma = -5$. B1 for the sum of the roots, B1 for the product. Skipping the divide is the standard way to lose both marks. [2]

A2 $\alpha + \beta = 6$ and $\alpha\beta = 4$ straight from the coefficients. Then $1/\alpha + 1/\beta = (\alpha + \beta)/(\alpha\beta) = 6/4 = 3/2$. B1 for the sum and product, M1 for the common denominator, A1 for 3/2. Combine over a common denominator and both known quantities appear. [3]

SECTION B · Standard

B1 $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 36 - 8 = 28$, and $\alpha^2\beta^2 = (\alpha\beta)^2 = 16$. A quadratic with those roots is $z^2 - 28z + 16 = 0$, that is z^2 minus (sum)z plus product, with the symmetric functions doing all the work. M1 for $(\alpha + \beta)^2 - 2\alpha\beta$, A1 for 28, B1 for $\alpha^2\beta^2 = 16$, A1 for the equation. [4]

B2 $2^3 - 16 + 2 + 6 = 0$, so $z - 2$ is a factor and the cubic is $(z - 2)(z^2 - 2z - 3) = (z - 2)(z - 3)(z + 1)$, roots 2, 3, -1. Sum: $4 = -(-4)$; product: $-6 = -6/1$. M1 for substituting $z = 2$, A1 for the value zero, M1 for factorising the cubic, A1 for the other two roots, B1 for the check. Both relations hold, a worthwhile check on the factorising. [5]

B3 Substitute $w = z + 1$, so $z = w - 1$: $(w - 1)^2 - 6(w - 1) + 4 = w^2 - 8w + 11 = 0$. Alternatively the new sum is $6 + 2 = 8$ and the new product $\alpha\beta + \alpha + \beta + 1 = 4 + 6 + 1 = 11$. Both routes give $z^2 - 8z + 11 = 0$. M1 for the substitution $w = z + 1$, dM1 for expanding, A1 for the new sum 8, A1 for the new product 11. [4]

B4 Read the coefficients off with alternating signs, as $z^3 - (\text{sum})z^2 + (\text{pairs})z - (\text{product})$. That gives $z^3 + 0z^2 - 2z + 8$, that is $z^3 - 2z + 8 = 0$. B1 for the alternating signs, M1 for inserting the three values, A1 for the cubic. The relations run both ways, from roots to coefficients as directly as from coefficients to roots. [3]

SECTION C · Stretch

C1 From the coefficients, $\Sigma\alpha = 4$, $\Sigma\alpha\beta = 1$ and $\alpha\beta\gamma = -6$. Squaring the first, $(\Sigma\alpha)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta$, so $\Sigma\alpha^2 = 16 - 2 =$ **14**. For the cubes, use the equation itself. Each root satisfies $\alpha^3 = 4\alpha^2 - \alpha - 6$, so summing over the three roots gives $\Sigma\alpha^3 = 4\Sigma\alpha^2 - \Sigma\alpha - 18 = 56 - 4 - 18 =$ **34**. For the new cubic, its three symmetric functions are $\Sigma\alpha^2 = 14$, $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2(\Sigma\alpha)(\alpha\beta\gamma) = 1 + 48 = 49$, and $\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 36$. So the equation is $\mathbf{z^3} - 14z^2 + 49z - 36 = 0$. B1 for the three symmetric functions, M1 for squaring the sum, A1 for 14, M1 for using the equation on each root, A1 for 34, M1 for $\Sigma\alpha^2\beta^2$, A1 for 49, B1 for the product 36, A1 for the cubic. The actual roots are 2, 3 and -1, whose squares 4, 9 and 1 do satisfy it, but the marks here are for the symmetric-function route. Squaring each coefficient of the original cubic is a common and completely invalid shortcut. [9]