inkMaths

ANSWER BOOK

Roots of unity and complex roots

Complex numbers · Year FM

7 questions · 31 marks

NAVY EDITION

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SECTION A · Warm-up

A1 Exactly five, the fifth roots of 32. Each has modulus $32^{1/5} = 2$; B1 for five solutions, B1 for the modulus. they differ only in argument, spaced $2\pi/5$ apart round the circle of radius 2. [2]

A2 Arguments $0, \pi/3, 2\pi/3, \pi, -2\pi/3, -\pi/3$, which mark a regular hexagon on the unit circle with one vertex at 1. Their sum is 0, by the rotational symmetry of the set. B1 for the six arguments, B1 for the regular hexagon, B1 for the sum being zero. [3]

SECTION B · Standard

B1 -27 has modulus 27 and argument π, so one root has modulus 3 and argument $\pi/3$, that is $z = 3(\cos \pi/3 + i \sin \pi/3) = 3/2 + (3\sqrt{3}/2)i$. The others sit $2\pi/3$ round, at $z = -3$ (argument π) and the conjugate $3/2 - (3\sqrt{3}/2)i$. M1 for the modulus and argument of -27, A1 for the first root, M1 for spacing the others $2\pi/3$ apart, A1 A1 for the remaining roots. Three roots, an equilateral triangle with the real root at -3. [5]

B2 Modulus 16 and argument π, so each root has modulus 2 with arguments $\pi/4$, $3\pi/4$, $-3\pi/4$, $-\pi/4$, giving $z = \pm\sqrt{2} \pm \sqrt{2}\, i$, all four sign combinations. They form a square of circumradius 2 with vertices on neither axis, one in each quadrant. M1 for modulus 2 and argument $\pi/4$, A1 for the four arguments, M1 for converting to $a + bi$, A1 for the four roots, B1 for the square. [5]

B3 $z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$. Any root other than 1 kills the second bracket, so the four non-trivial roots satisfy $z^4 + z^3 + z^2 + z + 1 = 0$; adding 1's contribution, the full sum of all five roots is the negated z^4-coefficient of the quintic, which is 0. M1 for the factorisation, A1 for the quartic satisfied by the other four roots, A1 for the sum being zero. [3]

B4 $64i$ has modulus 64 and argument $\pi/2$, so one root has modulus 4 and argument $\pi/6$, that is $z = 4(\cos \pi/6 + i \sin \pi/6) = 2\sqrt{3} + 2i$. Spacing the rest $2\pi/3$ apart gives arguments $5\pi/6$ and $-\pi/2$, so $z = -2\sqrt{3} + 2i$ and $z = -4i$. M1 for the modulus and argument of $64i$, A1 for the first root, M1 for spacing the others $2\pi/3$ apart, A1 A1 for the remaining roots. Checking the last one, $(-4i)^3 = -64i^3 = 64i$. [5]

SECTION C · Stretch

C1 w is a fifth root of unity other than 1, since $w^5 = \cos 2\pi + i \sin 2\pi = 1$ and $w \neq 1$. [8]
Factorising, $w^5 - 1 = (w - 1)(1 + w + w^2 + w^3 + w^4) = 0$, and $w - 1 \neq 0$, so the second bracket vanishes. By de Moivre, $w^k = \cos(2k\pi/5) + i \sin(2k\pi/5)$. Taking real parts of the sum gives $1 + \cos(2\pi/5) + \cos(4\pi/5) + \cos(6\pi/5) + \cos(8\pi/5) = 0$. Now $\cos(6\pi/5) = \cos(4\pi/5)$ and $\cos(8\pi/5) = \cos(2\pi/5)$, because w^3 and w^4 are the conjugates of w^2 and w, so the sum is $1 + 2\cos(2\pi/5) + 2\cos(4\pi/5) = 0$, giving $\mathbf{\cos(2\pi/5)} + \cos(4\pi/5) = -1/2$. B1 for $w^5 = 1$ with $w \neq 1$, M1 for the factorisation, A1 for the bracket vanishing, M1 for de Moivre on each power, A1 for the real part, M1 for pairing the conjugate cosines, A1 for the equation in two cosines, A1 for the printed result. Pairing the roots with their conjugates is what halves the work; the imaginary parts cancel in the same pairs.