inkMaths

ANSWER BOOK

Solving differential equations

Integration · Year 12–13

6 questions · 28 marks

NAVY EDITION

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SECTION A · Warm-up

A1 Multiply through by y and integrate both sides to reach $\int y\,dy = \int x^2\,dx$. [2]
Everything in y moves left with dy, everything in x stays right with dx. M1 for separating the variables, A1 for the two integrals. The separation is bookkeeping, and the calculus starts once it is done.

A2 Separate to get $\int (1/y)\,dy = \int 3x^2\,dx$, so $\ln y = x^3 + c$. Exponentiating gives $y = Ae^{x^3}$, where $A = e^c$. The condition gives $A = 2$, so $y = 2e^{x^3}$. M1 for separating, A1 for $\ln y = x^3 + c$, M1 for exponentiating, A1 for $y = 2e^{x^3}$. One constant, absorbed once and pinned once. Writing $y = e^{x^3} + c$ instead of absorbing the constant into a multiplier is the error that costs the accuracy marks. [4]

SECTION B · Standard

B1 Separate to get $\int y^{-2}\,dy = \int dx$, so $-1/y = x + c$. The condition gives $c = -1$, and rearranging gives $y = 1/(1 - x)$. The solution grows without bound as $x \to 1$, so it is valid only for $x < 1$. M1 for separating, A1 for $-1/y = x + c$, M1 for using $y = 1$ at $x = 0$, A1 for $y = 1/(1 - x)$, B1 for $x < 1$. A tame-looking equation has produced a vertical asymptote, and the validity statement carries its own mark. [5]

B2 Separate to get $\int h^{-1/2}\,dh = -k\int dt$, so $2\sqrt{h} = -kt + c$. The condition gives $c = 8$, hence $\sqrt{h} = (8 - kt)/2$ and $h = (8 - kt)^2/4$. The depth reaches zero when $kt = 8$, that is at $t = 8/k$. M1 for separating, A1 for $2\sqrt{h} = -kt + c$, M1 for $c = 8$, A1 for $h = (8 - kt)^2/4$, A1 for $t = 8/k$. Unlike an exponential decay, this tank genuinely empties in finite time. Note that the model only describes the drain up to $t = 8/k$, since beyond that the formula would have the depth rising again. [5]

B3 Separating and integrating gives $\ln V = -kt + c$, so $V = 500e^{-kt}$ once the initial condition is applied. Then $250 = 500e^{-10k}$ gives $e^{-10k} = ½$, so $10k = \ln 2$ and $k = (\ln 2)/10 = 0.0693$ per minute. M1 for separating, A1 for $\ln V = -kt + c$, A1 for $V = 500e^{-kt}$, M1 for $250 = 500e^{-10k}$, A1 for $k = 0.0693$. The ten-minute halving is this model's half-life, and every further ten minutes halves the volume again. Taking logarithms of ½ and mislaying the minus sign is the usual route to a negative k, which no leaking tank can have. [5]

SECTION C · Stretch

C1 Separate to get $\int 1/[P(5 - P)]\,dP = \int 0.1\,dt$. The left-hand integrand needs partial fractions: $1/[P(5 - P)] = (1/5)[1/P + 1/(5 - P)]$. Integrating gives $(1/5)[\ln P - \ln(5 - P)] = 0.1t + c$, so $\ln[P/(5 - P)] = 0.5t + 5c$. The condition $P = 1$ at $t = 0$ gives $\ln(1/4) = 5c$, so $\ln[P/(5 - P)] = 0.5t - \ln 4$ and $P/(5 - P) = \frac{1}{4}e^{0.5t}$. Cross-multiplying, $4P = (5 - P)e^{0.5t}$, so $P(4 + e^{0.5t}) = 5e^{0.5t}$ and $P = 5e^{0.5t}/(4 + e^{0.5t})$. Dividing top and bottom by $e^{0.5t}$ gives the required form. M1 for separating, M1 for partial fractions, A1 for the correct pair, A1 for the integrated form, M1 for using $P = 1$ at $t = 0$, A1 for $P/(5 - P) = \frac{1}{4}e^{0.5t}$, A1 for the printed form. Three places lose marks. The $1/(5 - P)$ term integrates to $-\ln(5 - P)$, not $+\ln(5 - P)$, because of the inner coefficient of -1. The factor of $1/5$ must be cleared before exponentiating. And the final division by $e^{0.5t}$ is needed to match the printed answer, so stopping one line early forfeits the last mark on a show-that. [7]