inkMaths

ANSWER BOOK

Systems of equations and invariance

Matrices · Year FM

7 questions · 31 marks

NAVY EDITION

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SECTION A · Warm-up

A1 M has rows (2, 1) and (1, 3), v is the column (x, y) and b the column (7, 11). B1 B1 B1 for M, v and b. One matrix equation carries both scalar equations at once. [3]

A2 det M = 6 - 1 = 5, so M^{-1} = (1/5) × rows (3, -1), (-1, 2). Then v = M^{-1}b = (1/5)(21 - 11, -7 + 22) = (2, 3). Check: 2 × 2 + 3 = 7 and 2 + 9 = 11. M1 for det M = 5, A1 for the inverse, M1 for v = M^{-1}b, A1 for (2, 3). [4]

SECTION B · Standard

B1 Either a prism, where the three pairwise intersection lines are parallel and the planes fence off a triangular tube, or two planes meeting in a line that the third misses everywhere by being parallel to it. The prism is the configuration with three parallel intersection lines; both give an inconsistent system with a singular coefficient matrix. B1 B1 for the two configurations, B1 for naming the prism, B1 for the singular coefficient matrix. [4]

B2 Take (1, 1) on the line. It maps to (5, 5), still on y = x, and any multiple (t, t) maps to (5t, 5t) by linearity. The line maps into itself with stretch factor 5. M1 for mapping a general point of the line, A1 for the image (5t, 5t), A1 for the conclusion, B1 for the stretch factor 5. This is an invariant line rather than a line of fixed points. Every point on it moves, but none of them leaves the line. [4]

B3 Try y = mx. The image of (1, m) is (4 + m, 2 + 3m), which lies on y = mx when 2 + 3m = m(4 + m), so $m^2 + m - 2 = 0$, giving m = 1 or m = -2. The new line is y = -2x, stretched by factor 2, since (1, -2) maps to (2, -4). M1 for trying y = mx, A1 for $m^2 + m - 2 = 0$, A1 for y = -2x, B1 for the stretch factor 2. [4]

B4 On an invariant line each point maps to a point of the same line, possibly elsewhere; a line of invariant points maps every point to itself. For a reflection, the mirror line is a line of invariant points, while any line perpendicular to the mirror is invariant without its points being fixed. B1 B1 for the two definitions, B1 B1 for the mirror line and a perpendicular line as the two examples. [4]

SECTION C · Stretch

C1 Expanding along the top row, det $M = (2k - 9) - (k - 6) + (3 - 4) = k - 4$, so M is singular when $k =$ **4**. For any other k the inverse exists and there is a single point of intersection. With $k = 4$ the third equation reads $2x + 3y + 4z = 5$, which is exactly the sum of the first two, so it carries no new information and the system is **consistent**. Solving the first two, subtracting gives $y + 2z = 3$, so $y = 3 - 2z$ and then $x = 1 - y - z = -2 + z$. Writing $z = \lambda$, the solutions form the line $r = (-2, 3, 0) + \lambda(1, -2, 1)$. Geometrically the three planes form a sheaf, meeting in that common line. M1 for expanding the determinant, A1 for det $M = k - 4$, A1 for $k = 4$, M1 for testing consistency, A1 for the third equation being the sum of the first two, M1 for solving the first two, A1 for the line of solutions, B1 for the sheaf. A singular matrix rules out a unique solution but says nothing about consistency; the sheaf here and an inconsistent prism both give det $M = 0$, so the equations still have to be tested. [8]