inkMaths

ANSWER BOOK

Taylor series

Further Pure 1 · Year FM

7 questions · 30 marks

NAVY EDITION

SECTION A · Warm-up

A1 $f(x) = f(a) + f'(a)(x - a) + f''(a)(x - a)^2/2! + \ldots$. B1 for the linear term, B1 for the quadratic term with its factorial. Each coefficient is the matching derivative evaluated at a, divided by the factorial of that power. [2]

A2 Every derivative of the exponential equals itself, so all the values at $x = 1$ are e. The series is $e + e(x - 1) + e(x - 1)^2/2 + \ldots$, which is e times the standard exponential series shifted along. B1 for every derivative being e at $x = 1$, M1 for substituting into the Taylor form, A1 for the series. [3]

SECTION B · Standard

B1 Derivatives at 1: $\ln 1 = 0$, $1/x = 1$, $-1/x^2 = -1$, $2/x^3 = 2$. So $\ln x = (x - 1) - (x - 1)^2/2 + (x - 1)^3/3 - \ldots$. At $x = 1.1$: $0.1 - 0.005 + 0.000333 = 0.09533$, against the true 0.09531. M1 for differentiating three times, A1 A1 for the derivative values, A1 for the series, B1 for 0.09533. Three terms give four decimal places, because the anchor sits so close. [5]

B2 At $\pi/4$: $\tan = 1$, $\sec^2 = 2$, and the second derivative $2\sec^2 x \tan x = 4$. So $\tan x = 1 + 2(x - \pi/4) + 2(x - \pi/4)^2 + \ldots$. At the stated point: $1 + 0.2 + 0.02 = 1.22$, against the true 1.2230. M1 for differentiating twice, A1 for 2, A1 for 4, A1 for the series, B1 for 1.22. Correct to 2 decimal places from a quadratic. [5]

B3 Every term after the first carries a power of $(x - a)$, which is small near the anchor, so the neglected terms are tiny there. A Maclaurin series carries powers of x instead, which are large when x is far from the origin, so far more terms are needed for the same accuracy. B1 for the powers of $(x - a)$ being small near the anchor, B1 for the neglected terms being tiny there, B1 for powers of x being large far from the origin. [3]

B4 Derivatives at 9: $\sqrt{9} = 3$, $1/(2\sqrt{x}) = 1/6$, and $-1/(4x^{3/2}) = -1/108$. So $\sqrt{x} = 3 + (x - 9)/6 - (x - 9)^2/216 + \ldots$. At $x = 9.6$: $3 + 0.1 - 0.001667 = 3.098$, against the true 3.0984. M1 for differentiating twice, A1 for 1/6, A1 for -1/108, A1 for the series, B1 for 3.098. Anchoring at the nearest perfect square is what makes the arithmetic this light. [5]

SECTION C · Stretch

C1 Differentiate the equation repeatedly, substituting the known values as you go. [7]
At $x = 1$, $y = 1$ and $dy/dx = 1 + 1 = 2$. Differentiating $dy/dx = x + y^2$ gives $d^2y/dx^2 = 1 + 2y(dy/dx)$, which at $x = 1$ is $1 + 2(1)(2) = 5$. Differentiating again, and using the product rule on the $2y(dy/dx)$ term, gives $d^3y/dx^3 = 2(dy/dx)^2 + 2y(d^2y/dx^2)$, which at $x = 1$ is $2(4) + 2(1)(5) = 18$. The Taylor series is $y = 1 + 2(x - 1) + 5(x - 1)^2/2! + 18(x - 1)^3/3!$, that is $\mathbf{y} = 1 + 2(x - 1) + (5/2)(x - 1)^2 + 3(x - 1)^3 + \dots$. B1 for $dy/dx = 2$, M1 for differentiating the equation, A1 for 5, M1 for the product rule at the next stage, A1 for 18, M1 for the Taylor form with its factorials, A1 for the series. Two errors dominate here. Forgetting the factorial denominators turns 18 into the coefficient rather than 3, and differentiating y^2 as $2y$ instead of $2y(dy/dx)$ loses the chain rule that makes the recursion work.